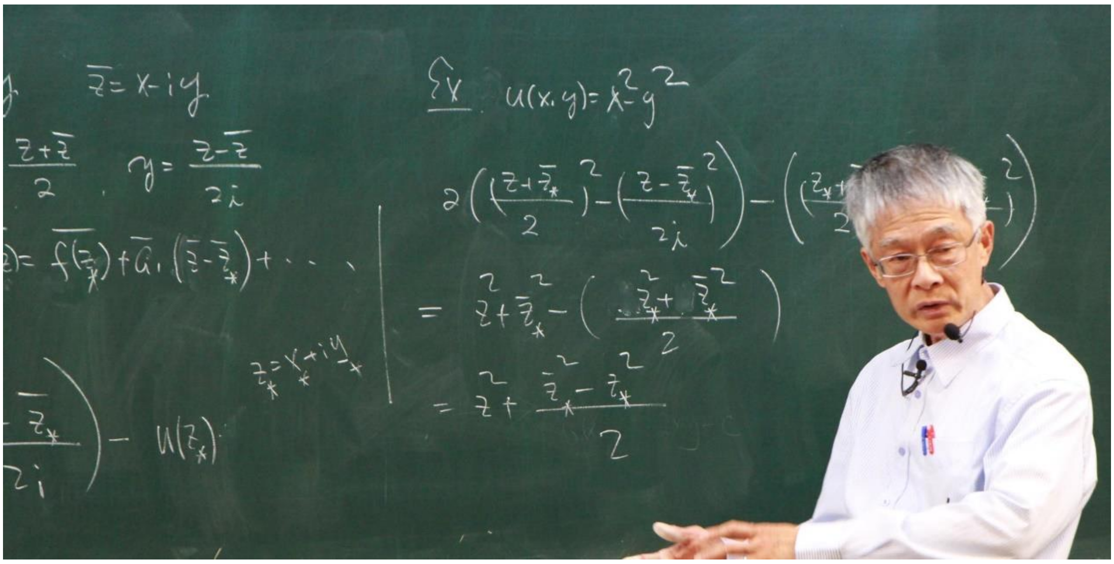
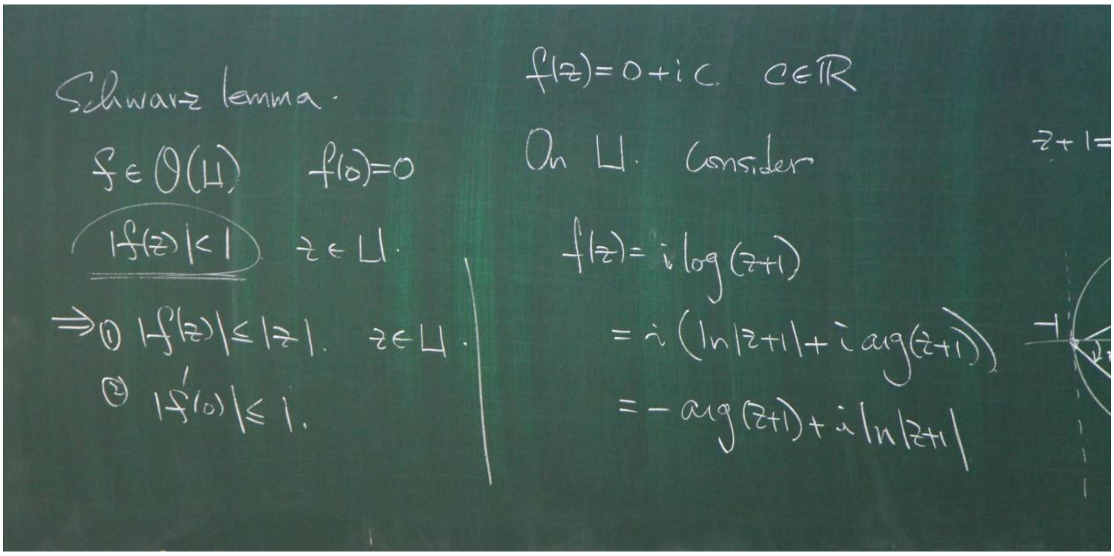
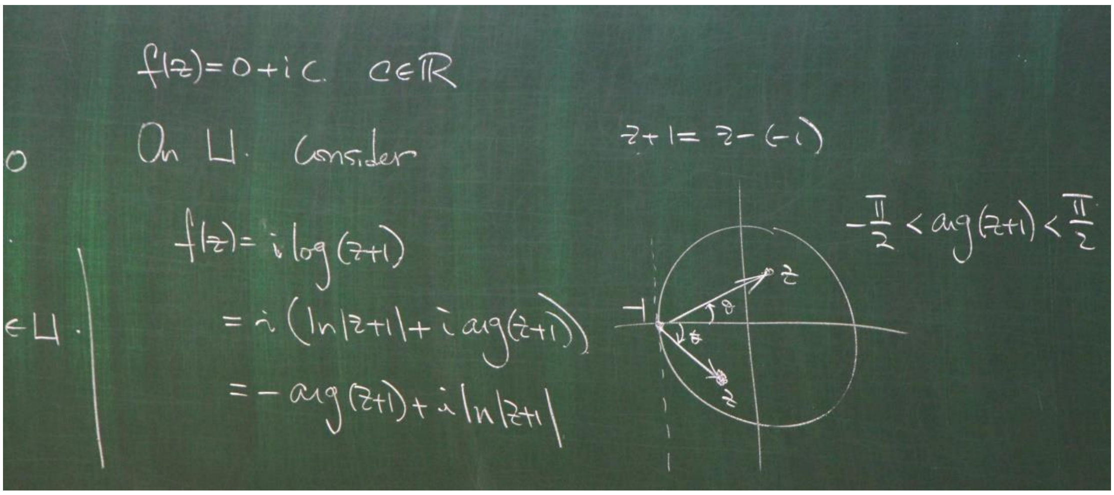
【10920 程守慶教授複變數函數論 / 第 14 堂版書】







Thm:

Suppose $f \in \mathcal{O}(\mathbb{C})$, and

Re f is bounded on $\mathbb{C}$.

$\Rightarrow$ f is a constant function.

Set $r = \frac{R}{2}$. R: large.

For $|z| \le r$

f(0))

$$|f(z)| \le |f(0)| + \frac{R}{\frac{R}{2}}\left(A(R) - \mathrm{Re}f(0)\right) = |f(0)| + 2\left(A(R) - \mathrm{Re}f(0)\right) < M < \infty$$

$\in\mathcal{O}(\mathbb{C})$, and.

is bounded on $\mathbb{C}$.

a constant function

Case(ii) in general

set $h(z) = f(z) - f(0)$. $h(0) = 0$

$$|h(re^{i\theta})| \le \frac{2r}{R-r} A_h(R)$$

$$f(z) = h(z) + f(0).$$

Case(i) Assume $f(0) =$

show $|f(re^{i\theta})| \le$

$$|f(re^{i\theta})| \le |f(0)| + |h(re^{i\theta})| \le |f(0)| + \frac{2r}{R-r}\Big(\underbrace{\max_\theta \mathrm{Re}f(Re^{i\theta})}_{} - \mathrm{Re}f(0)\Big)$$

$$\max_\theta \mathrm{Re}f(Re^{i\theta}) = A(R)$$

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$$|f(re^{i\theta})| \le |f(0)| + |h(re^{i\theta})| \le |f(0)| + \frac{2r}{R-r}\Big(\max_\theta \mathrm{Re}f(Re^{i\theta}) - \mathrm{Re}f(0)\Big)$$

$$\max_\theta \mathrm{Re}f(Re^{i\theta}) = A(R)$$

Case(i) Assume $f(0)=0$

show $|f(re^{i\theta})| \leq \dfrac{2rA(R)}{R-r}$

Assume f is not a constant function

$\Rightarrow A(R) > 0$.

Consider $|z| < r < R$.

$$g(z) = \frac{f(z)}{2A(R) - f(z)} = \frac{f(z)}{2A(R) - \operatorname{Re} f(z) - i \operatorname{Im} f(z)}$$

$f(Re^{i\theta}) - \operatorname{Re} f(0)$

$$|g(z)|^2 \leq \frac{\overset{\operatorname{Re}^2 f(z) + \operatorname{Im}^2 f(z)}{\overset{\shortparallel}{|f(z)|^2}}}{(2A(R) - \operatorname{Re} f(z))^2 + (\operatorname{Im} f(z))^2} < \frac{|f(z)|^2}{\operatorname{Re}^2 f(z) + \operatorname{Im}^2 f(z)} = 1.$$

$$\leq \frac{\overset{\operatorname{Re}^2 f(z) + \operatorname{Im}^2 f(z)}{\overset{\shortparallel}{|f(z)|^2}}}{(2A(R) - \operatorname{Re} f(z))^2 + (\operatorname{Im} f(z))^2} < \frac{|f(z)|^2}{\operatorname{Re}^2 f(z) + \operatorname{Im}^2 f(z)} = 1.$$

$\operatorname{Im} f(z)$

$g : \{|z| < R\} \longrightarrow \mathbb{C}$

$$\begin{tikzcd} \mathbb{U} \arrow[r] \arrow[rr, bend right, "h"] & \{|z|<R\} \arrow[r, "g"] & \mathbb{C} \end{tikzcd}$$

$z \longmapsto Rz$

$h : \mathbb{U} \to \mathbb{U}$ $\quad h(0) = 0$ $\quad$ Schwarz lemma

$z \mapsto g(Rz)$ $\quad \boxed{|h(z)| \leq |z|}$

Harmonic functions.

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$f \in C^2$

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ Laplace operator

Laplacian.

satisfies.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 = \Delta f$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

<u>Complex structure</u> $J : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\mathbb{R}$-linear 1-1, onto.

s.t. $J^2 = J \circ J = -\mathrm{id}$.

<u>Ex</u>

$\mathbb{R}^2$, $\mathbb{C}$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$J^2 = ($

$= ($

$= -$

$J: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

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<u>Ex</u>

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Complex one-dim complex vector space

$(\mathbb{R}^2, J) \simeq \mathbb{C}$

Define, $x \in \mathbb{R}^2$,

$ix = Jx$

$D \subseteq \mathbb{C}$ domain

$f \in \mathcal{O}(D)$

$\partial \bar{\partial} f = 0$

$\| $

$\frac{1}{4} \Delta f$

$\mathrm{Re} f$, $\mathrm{Im} f$

harmonic on D

Given a $\overset{\text{real}}{\text{harmonic}}$ ft. u on D.

does there exist a $\overset{\text{(real)}}{\text{harmonic}}$ ft v on D.

s.t. $u + iv \in \mathcal{O}(D)$?

harmonic conjugate

$D = \{0 < |z| < 1\}$, punctured disc

$u(x, y) = \ln(x^2 + y^2)^{1/2}$

place operator

placian.

$\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.

$D \subseteq \mathbb{C}$ domain

$f \in \mathcal{O}(D)$

$\partial \bar{\partial} f = 0$

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$\frac{1}{4} \Delta f$

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Imf

nic on D

real

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$\uparrow$

harmonic conjugate

$D = \{0 < |z| < 1\}$, punctured disc

$u(x,y) = \ln(x^2 + y^2)^{1/2}$

<u>locally</u>, $u+iv$ is <u>holomorphic</u> $\Leftrightarrow$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$\therefore\ v(x,y) = \overset{g(x,y)}{\overset{\|}{\boxed{\int u_x(x,y)\,dy}}} + \phi(x)$$

$$v_x(x,y) = -u_y(x,y) = g_x(x,y) + \phi'(x)$$

<u>Ex</u>. $u(x,y)=$

$\Delta u=$

$u_x = 2x$

$\int 2x\,dy +$

$2xy +$

$2y +$

<u>Ex</u>. $u(x,y) = x^2 - y^2$

$\Delta u = 2-2=0$

$u_x = 2x,\quad u_y = -2y.$

$$\int 2x\,dy + \phi(x) = v(x,y)$$

$$2xy + \phi(x) = v(x,y) \longrightarrow v(x,y) = 2xy + C.$$

$$\cancel{2y} + \phi'(x) = v_x(x,y) = \cancel{2y}$$

$\phi'(x) = 0.$

$\phi(x) =$ constant $= C \in \mathbb{R}$.

$$\Longrightarrow \quad u(x,y) + iv(x,y) = x^2 - y^2 + i(2xy + c) = x^2 - y^2 + i2xy + ic. = z^2 + iC$$

Harmonic functions.

$D \subseteq \mathbb{C}$ domain.

⊕ D is simply-connected

let u be a real harmonic ft on D

$\Rightarrow$ $v(x,y) = \int_{z_0}^{z} -u_y(s,t)\,ds + u_x(s,t)\,dt$

γ: a curve from z_0 to z

is a har

pf.

is a harmonic conjugate of u on D.

pf. $\int_\gamma = \int_T \Leftrightarrow \int_{\gamma - T} = 0$

$$\int_{\gamma - T} -u_y(s,t)\,ds + u_x(s,t)\,dt \overset{\text{Green}}{=} \iint_\Omega (u_{xx}+u_{yy})\,dx\,dy = 0$$

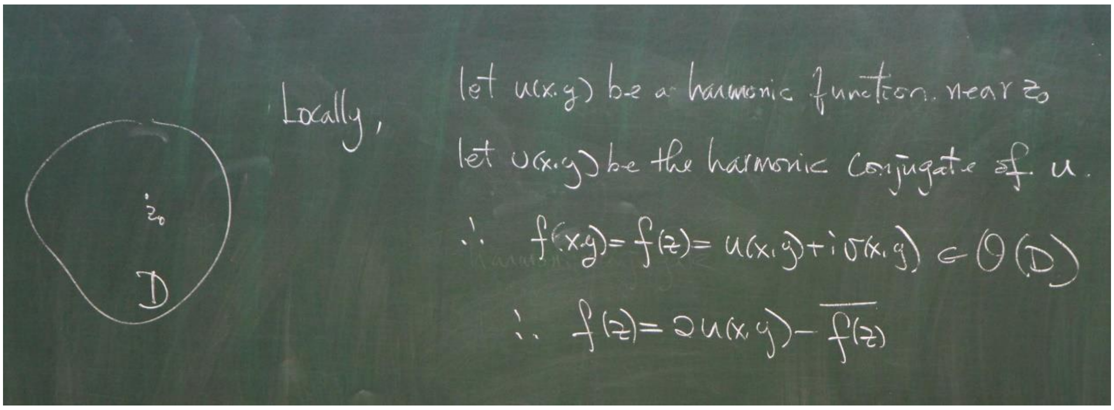
Locally,

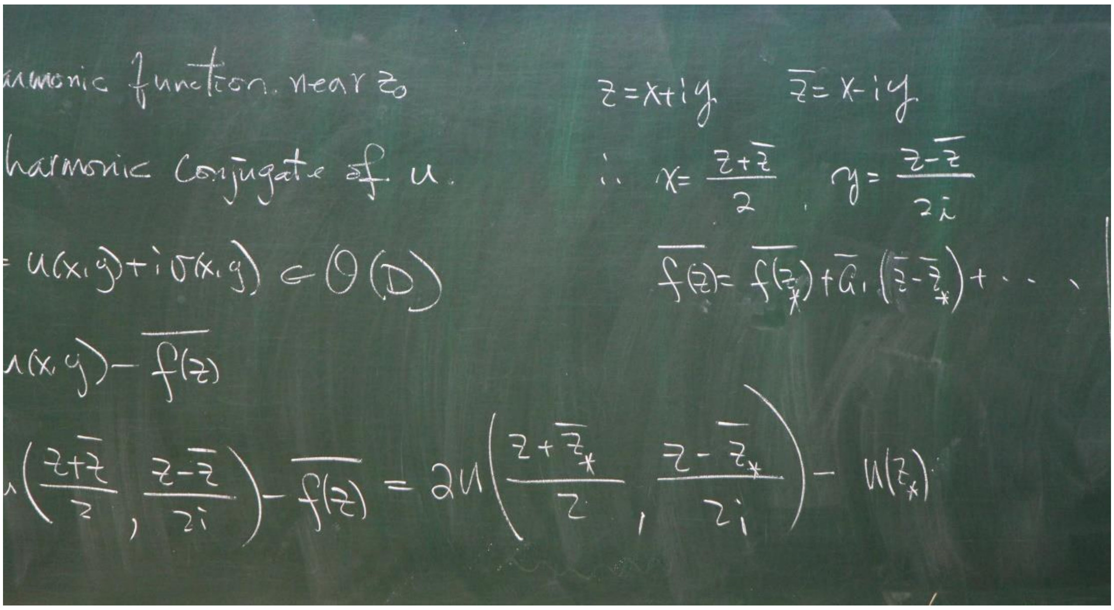
let $u(x,y)$ be a harmonic functi

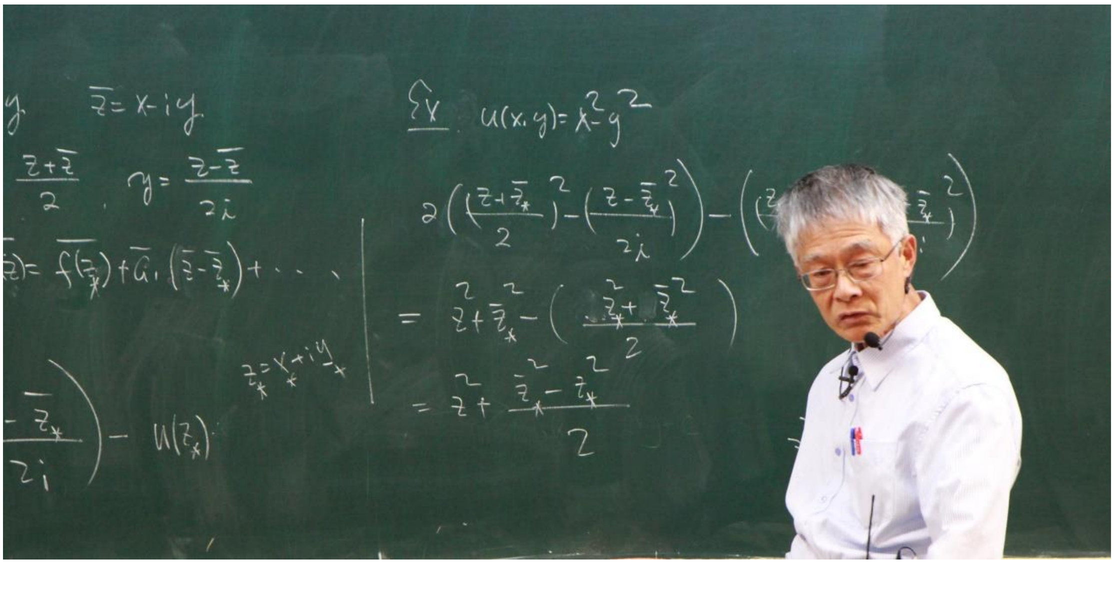
let $v(x,y)$ be the harmonic conj

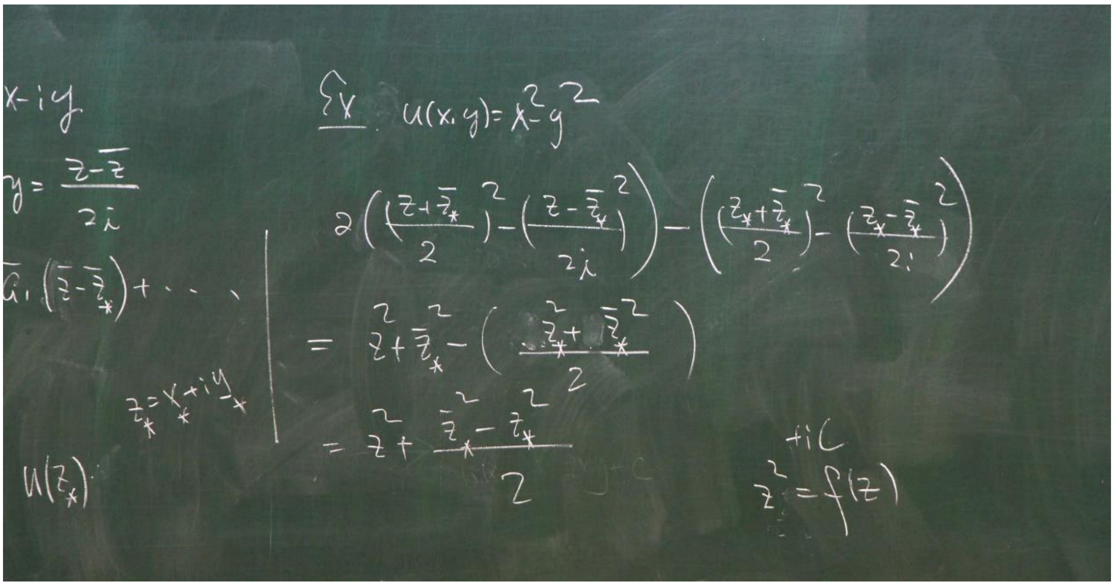
$\therefore$ $f(x,y) = f(z) = u(x,y) + i\,v(x,$

$\therefore$ $f(z) = 2u(x,y) - \overline{f(z)}$


Locally,
let $u(x,y)$ be a harmonic function near z_0
let $v(x,y)$ be the harmonic conjugate of u.
$\therefore\ f(x,y)=f(z)=u(x,y)+iv(x,y)\in\mathcal{O}(D)$
$\therefore\ f(z)=2u(x,y)-\overline{f(z)}$
z_0
D


$z=x+iy$ $\quad \bar z=x-iy$
$\therefore\ x=\frac{z+\bar z}{2},\ y=\frac{z-\bar z}{2i}$
$\overline{f(z)}=\overline{f(z_*)}+\bar a_1(\bar z-\bar z_*)+\cdots,$
$u\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2i}\right)-\overline{f(z)}=2u\left(\frac{z+\bar z_*}{2},\frac{z-\bar z_*}{2i}\right)-u(z_*)$


Ex. $u(x,y)=x^2-y^2$
$2\left(\left(\frac{z+\bar z_*}{2}\right)^2-\left(\frac{z-\bar z_*}{2i}\right)^2\right)-\left(\cdots\right)$
$=z^2+\bar z_*^2-\left(\frac{z_*^2+\bar z_*^2}{2}\right)$
$=z^2+\frac{\bar z_*^2-z_*^2}{2}$
$z_*=x_*+iy_*$


x - iy
Ex
u(x, y) = x^2 - y^2
u(z_*)
z^2 = f(z)
+iC